Lattice polarized toric K3 surfaces

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Abstract

When studying mirror symmetry in the context of K3 surfaces, the hyperkähler structure of K3 makes the notion of exchanging Kähler and complex moduli ambiguous. On the other hand, the metric is not renormalized due to the higher amount of supersymmetry of the underlying superconformal field theory. Thus one can define a natural mapping from the classical K3 moduli space to the moduli space of conformal field theories. Apart from the generalization of mirror constructions for Calabi-Yau threefolds, there is a formulation of mirror symmetry in terms of orthogonal lattices and global moduli space arguments. In many cases both approaches agree perfectly — with a long outstanding exception: Batyrev's mirror construction for K3 hypersurfaces in toric varieties does not fit into the lattice picture whenever the Picard group of the K3 surface is not generated by the pullbacks of the equivariant divisors of the ambient toric variety. In this case, not even the ranks of the corresponding Picard lattices add up as expected. In this paper the connection is clarified by refining the lattice picture. We show (by explicit calculation with a computer) mirror symmetry for all families of toric K3 hypersurfaces corresponding to dual reflexive polyhedra, including the formerly problematic cases.

1 Introduction

Superstring theory at small coupling is described by superconformal field theories on the string world sheet at central charge $c = 15$. The most simple theory of this kind consists of 10 free bosons (each contributing 1 to the central charge) and 10 fermions (contributing central charge $1/2$) leading to string theory in flat 10 dimensional Minkowski space. In order to obtain theories with fewer (visible) space-time dimensions[2], one considers products of (10-2D) dimensional flat Minkowski space-time theories with internal superconformal field theories with central charge $c = 3D$.

Under certain assumptions[3] one expects all components of the moduli space of such theories to contain boundaries consisting of supersymmetric sigma models on Ricci-flat Kähler manifolds X of large radius. This allows the use of

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[2] We do not consider the more modern approach of confining (most of the) fields to lower dimensional branes.

[3] if the conformal field theories contain the spectral flow operator in their Hilbert space

classical geometrical methods for studying these theories. In the (realistic) case of four flat space-time dimensions X is a Calabi-Yau threefold. This intimate connection between quantum field theories and classical geometry led to one of the most striking predictions concerning the classical geometries — mirror symmetry: The deformations of the classical geometry fall into two seemingly unrelated classes, namely the deformations of the complex structure labelled by the cohomology group $H^{D-1,1}(X)$ and deformations of the complexified Kähler form[4] labelled by $H^{1,1}(X)$.

From the viewpoint of conformal field theory these deformations correspond to marginal operators, which also fall into two classes, namely $h = \bar{h} = \frac{1}{2}$, $q = \bar{q} = 1$ and $h = \bar{h} = \frac{1}{2}$, $q = -\bar{q} = 1$. h denotes the conformal dimension of the marginal operator and q its charge under the $U(1)$ current contained in the superconformal Virasoro algebra. Although we also have to deal with two kinds of operators, the difference seems much less fundamental than for their geometric counterparts. In fact, changing the sign of the $U(1)$-charges leads to an isomorphic conformal field theory [GP90].

If this field theory also has a geometric interpretation, it should be a sigma model on another Calabi-Yau threefold $\tilde{X}$ with $H^{D-1,1}(\tilde{X}) = H^{1,1}(X)$ and vice versa.

This observation led to the remarkable conjecture, that Calabi-Yau manifolds should come in pairs with the Hodge groups exchanged (and, of course, isomorphic underlying superconformal field theories). Though at first counter-intuitive for mathematicians, lots of evidence for this conjecture has been found and it is nowadays widely accepted.

For large classes of Calabi-Yau manifolds recipes for constructing mirror partners are known, e.g. the quotient construction of [GP90], its generalization to hypersurfaces in toric varieties corresponding to reflexive polyhedra [Bat94], generalizations thereof [Bor93, BB94, B+98] and fiberwise T-Duality [SYZ96].

If one considers compactifications not to 4, but rather 6 (real) dimensions, the compactification space has to be either a real compact 4-torus or a K3 surface. Since K3 surfaces are hyperkähler manifolds (the underlying superconformal field theories possess a higher amount of supersymmetry[5]), the (local) factorization of the moduli space into complex and Kähler deformations is lost. Hence, the generalization of the above mirror symmetry to this case is not so obvious.

On the other hand, the higher supersymmetry implies nonrenormalization of the metric, which allows comparison of geometry and conformal field theory not only near a large radius limit. For the K3 case, the most beautiful approach to mirror symmetry is given by global moduli space arguments in terms of orthogonal Picard lattices [AM97, Vaf89, Dol96]. In fact, this description is a quantum geometrical version of a much older duality discovered by Dolgachev, Nikulin and (independently) Pinkham [DN77, Pin77] and used to

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[4]In the context of type II string theory one always has to include an antisymmetric B-field, which is only defined modulo integer classes and conveniently combined with the Kähler form to form the so-called complexified Kähler form.

[5]$N = (4, 4)$ instead of $N = (2, 2)$

explain Arnold's strange duality [Arn74]. I will give a short exposition of this picture in section 2.

As already remarked in [Dol96], this version of mirror symmetry fits in nicely with known mirror constructions. A subtlety arises in the case of Batyrev's mirror construction for toric hypersurfaces. The mirror symmetry picture drawn in [AM97, Dol96] fails in the case of nonvanishing toric correction term (the difference between the ranks of the Picard groups of the generic K3 hypersurface and the ambient toric variety). As these toric hypersurfaces are not generic members of the families corresponding to their Picard lattices, this does not imply failure of either picture. Nevertheless the failure to match both pictures in these cases is unsatisfactory. In [Dol96, Conj. 8.6] it was conjectured, that in order to obtain a matching the Picard lattice has to be replaced by a suitably chosen sublattice. The main purpose of this paper is to explicitly state the involved lattices and to prove (by computer) the mirror assertion for all reflexive polyhedra of dimension 3 — including the case of nonvanishing correction term.

The layout of the paper is as follows: In section 2 I will give a brief exposition of K3 mirror symmetry in terms of orthogonal lattices, including the necessary refinements to treat the case of toric hypersurfaces with nonvanishing toric correction. Section 3 is devoted to the calculation of the Picard lattice for toric K3 hypersurfaces. In section 4 I will state the main theorem concerning mirror symmetric families of toric K3 hypersurfaces and describe the algorithm used to prove it.

2 Moduli spaces and orthogonal lattices

2.1 Moduli space of superconformal field theories and geometric interpretations

The exposition in this section roughly follows [Nah00] which is based on [Wen00, NW01, AM97, Asp97].

We first recall some well known facts about the space of Ricci-flat Kähler metrics[6] on $X = K3$. Given such a metric (and an orientation on X) consider the action of the Hodge star operator $\star$ on the total cohomology. We have $\dim H^0(X, \mathbb{R}) = \dim H^4(X, \mathbb{R}) = 1$ while the odd cohomology groups vanish. The action of $\star$ on $H^2(X, \mathbb{R})$ does not depend on the scale of the metric but only on the conformal structure. The wedge product of two elements of $H^2(X, \mathbb{R})$ together with the identification $H^4(X, \mathbb{R}) \cong \mathbb{R}$ given by the standard generator yields a scalar product[7] with signature $(3, 19)$. Because of $\star^2 = 1$ on $H^2(X, \mathbb{R})$ it splits into eigenspaces $H^+ \oplus H^-$ with eigenvalues $+1$ and -1, respectively. H^+ has dimension 3 and is positive definite, while H^- has dimension 19 and is negative definite. The orientation on X induces an orientation on H^+. Since

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[6]As remarked before, for the description of the moduli space of K3 surfaces the distinction between complex structure and Kähler moduli does not make much sense.

[7]This is the same as the cup product or the intersection product on homology via Poincaré duality and the de Rham isomorphism.

one can show that Ricci flat metrics with fixed volume are locally uniquely specified by $H^+ \subset H^2(X, \mathbb{R})$, the tangent space to the corresponding moduli space is given by $so(H^2(X, \mathbb{R}))/(so(H^+) \oplus so(H^-))$. The Teichmüller space of metrics with fixed volume, including orbifold limits, is the Grassmannian

$$\mathcal{T}^{3,19} = O^+(3, 19)/(SO(3) \times O(19))$$

of oriented positive definite 3-planes $\Sigma \subset H^2(X, \mathbb{R})$. The right hand side gives the connection to the tangent space description above, where $O^+(3, 19)$ is the index 2 subgroup of $O(3, 19)$ containing $SO(3) \times O(19)$.

Now let $\Gamma(3, 19)$ be the intersection of $O^+(3, 19)$ with the automorphism group of[8] $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$. As $\Gamma(3, 19)$ is the diffeomorphism group of the K3 surface [Mat85, Bor86, Don90], the moduli space of metrics (again including orbifold limits) is

$$\Gamma(3, 19)\backslash \mathcal{T}^{3,19} \times \mathbb{R}^+ = \Gamma(3, 19)\backslash O^+(3, 19)/(SO(3) \times O(19)) \times \mathbb{R}^+,$$

where the extra $\mathbb{R}^+$ parametrizes the volume.

Turning to the space of superconformal field theories, it has long been conjectured [Sei88], that the duality group $\Gamma(4, 20)$ can be interpreted as automorphism group of the total integer cohomology $H^\star(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$. Correspondingly, the moduli space of superconformal field theories should be

$$\Gamma(4, 20)\backslash \mathcal{T}^{4,20} = \Gamma(4, 20)\backslash O^+(4, 20)/(SO(4) \times O(20)),$$

where $O^+(4, 20)$ should be interpreted as the orthogonal group of the total cohomology $H^0(X, \mathbb{R}) \oplus H^2(X, \mathbb{R}) \oplus H^4(X, \mathbb{R})$. Apart from $H^0(X, \mathbb{R}) \oplus H^4(X, \mathbb{R})$, all direct sums will be orthogonal.

In [AM97] Aspinwall and Morrison were able to construct the sigma model[9] (i.e. metric and B-field) for given positive definite 4-plane $\Xi \in \mathcal{T}^{4,20}$:

Let us fix an isomorphism $\mathbb{Z}(4, 20) \cong H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$ and let v, v^0 denote the standard generators of $H^4(X, \mathbb{Z})$ and $H^0(X, \mathbb{Z})$, respectively. Furthermore let $\Upsilon = \Xi \cap (H^2(X, \mathbb{R}) \oplus H^4(X, \mathbb{R}))$. Obviously, Υ is a positive definite 3-plane in $H^2(X, \mathbb{R}) \oplus H^4(X, \mathbb{R})$. Its projection to $H^2(X, \mathbb{R})$ yields the wanted positive definite 3-plane Σ. Σ is three dimensional because $(v)^2 = 0 \Rightarrow v \notin \Upsilon$ and positive definite because $v \perp H^2(X, \mathbb{R}) \oplus H^4(X, \mathbb{R})$. Inverting this projection, one can write

$$\Upsilon = \{\sigma - (B\sigma)v, \sigma \in \Sigma\},$$

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[8]For a K3 surface $H^2(X, \mathbb{Z})$ contains no torsion and therefore $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$.

[9]It is more natural to define the inverse mapping, i.e. to construct the fourplane for a given sigma model. Our approach just makes invertibility of the mapping more obvious.

One should note, that the existence of all sigma models in the image of the mapping has not been proven rigorously. However, the conformal dimensions and operator product expansion coefficients have a well behaved perturbation expansion in terms of inverse powers of the volume. We thus make the assumption that a rigorous treatment is possible.

where $B \in H^2(X, \mathbb{R})$ (the scalar product on $H^2(X, \mathbb{Z})$ is nondegenerate) and will be identified with the B-field. As yet, B is only specified up to elements of $\Sigma^\perp$.

If we write $\Xi = \Upsilon \oplus \mathbb{R}\xi$ and demand $\xi v = 1$, this uniquely determines ξ. Using the remaining freedom for the choice of B it can be written as

$$\xi = v^0 + B + (V - \frac{B^2}{2})v, \qquad V \in \mathbb{R}.$$

As Ξ is positive definite, we have $\xi^2 = 2V > 0$ and V can be identified with the volume. Obviously, the mapping $\Xi \overset{v,v^0}{\mapsto} (\Sigma, B, V)$ is invertible:

$$(\Sigma, B, V) \overset{v,v^0}{\mapsto} \langle \xi \rangle_\mathbb{R} \oplus \underline{\xi}(\Sigma)$$

where the vector ξ is defined as above and

$$\underline{\xi} : \begin{cases} H^2(X, \mathbb{R}) & \to & H^2(X, \mathbb{R}) \oplus H^4(X, \mathbb{R}) \\ \sigma & \mapsto & \sigma - (B\sigma)v \end{cases} .$$

The map $\mathcal{T}^{4,20} \to \mathcal{T}^{3,19}$ given by the construction indeed isomorphically maps the subgroup of $\Gamma(4, 20)$ leaving both v and v^0 invariant onto the classical automorphism group $\Gamma(3, 19)$.

Elements of $\Gamma(4, 20)$ leaving v invariant are symmetries of the superconformal field theory since they correspond to translations of B by an element of $H^2(X, \mathbb{Z})$.

In order to prove that the whole of $\Gamma(4, 20)$ yields isomorphic superconformal field theories, one has to prove equivalence for one additional generator, for which two choices have been made in the literature: T-duality by Nahm und Wendland [NW01] and mirror symmetry[10] by Aspinwall und Morrison [AM97].

We see that a given sigma model (Σ, B, V) specifies a conformal field theory, but the opposite direction depends on the choice of our sublattice[11] $H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z}) \cong H \subset \mathbb{Z}(4, 20)$, which is called a **geometric interpretation**[12].

2.2 Mirror symmetry

We now want to study mirror symmetry in this context. As mirror symmetry is an $N = (2, 2)$ phenomenon which in the case of Calabi-Yau threefolds involves splitting the moduli space's tangent space into complex and Kähler deformations, we have to be somewhat more specific than in the above discussion.

For a given oriented 4-plane Ξ (or the corresponding oriented 3-plane Σ), we have to choose a complex structure, which corresponds to choosing an oriented

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[10]The complete proof of this variant has never been published.

[11]together with the labelling of its isotropic subspaces as $H^0(X, \mathbb{Z})$ and $H^4(X, \mathbb{Z})$. The hyperbolic lattice H is defined to be $\mathbb{Z}^2$ with quadratic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

[12]It is also customary to call just the choice of v a geometric interpretation. As remarked above, this differs from our definition by still allowing for shifts of B by elements of $H^2(X, \mathbb{Z})$, which has no physical relevance.

2-plane $\Omega \subset \Sigma$ (defined by the real and imaginary part of the complex structure). This choice (together with the volume) also fixes the Kähler class compatible with the hyperkähler structure given by Σ.

In terms of cohomology the choice of Ω fixes $H^{2,0}(X,\mathbb{C}) \oplus H^{0,2}(X,\mathbb{C})$. The orthogonal complement of Ω in $H^2(X,\mathbb{R})$ then yields $H^{1,1}(X,\mathbb{R})$ and any vector $\omega \in H^{1,1}(X,\mathbb{R})$ of positive length $2V$ defines a Kähler class compatible with the complex structure given by Ω and the hyperkähler structure defined by the oriented 3-plane spanned by Ω and ω.

Equivalent to the choice of a 2-plane $\Omega \subset \Sigma$ is the choice of its lift $\tilde{\Omega} \subset \Upsilon \subset \Xi$. The latter corresponds to choosing a specific $N = (2,2)$-subalgebra within the $N = (4,4)$ superconformal algebra. More specifically, it corresponds to choosing a Cartan torus $u(1)_l \oplus u(1)_r$ of $su(2)_l \oplus su(2)_r$, where the rotations of Ξ in the twoplane $\tilde{\Omega}$ are generated by $u(1)_{l+r}$ and those in the orthogonal plane $\mho := \tilde{\Omega}^\perp \subset \Xi$ are generated by $u(1)_{l-r}$ ($SU(2)_l \times SU(2)_r$ operates on Ξ via the covering map $SU(2)_l \times SU(2)_r \to SO(4)$, for details c.f. [NW01, Wen00]).

The choice of $\tilde{\Omega}$ also induces a local splitting of the moduli space into deformations of $\mho \subset \tilde{\Omega}^\perp \subset H^\star(X,\mathbb{R})$ and deformations of $\tilde{\Omega} \subset \mho^\perp \subset H^\star(X,\mathbb{R})$.

We now want to specify what we mean by a K3 mirror symmetry. On the level of $N = (2,2)$ superconformal quantum field theories we know what we want to call a mirror symmetry, namely $u(1)_r \leftrightarrow -u(1)_r$ or equivalently $u(1)_{l+r} \leftrightarrow u(1)_{l-r}$. In our description, this corresponds to the interchange $\tilde{\Omega} \leftrightarrow \mho$ and leads to the following definition:

Definition 2.1 *Let $(\Xi = \tilde{\Omega} \oplus \mho, \mathbb{Z}v \oplus \mathbb{Z}v^0)$ be the superconformal field theory corresponding to (Σ, B, V) under the geometric interpretation given by v, v^0, together with a choice of an $N = (2,2)$-subalgebra given by $\tilde{\Omega}$. Further let $(\Xi' = \tilde{\Omega}' \oplus \mho', \mathbb{Z}v \oplus \mathbb{Z}v^0)$ be the theory corresponding to (Σ', B', V') with the choice of the $N = (2,2)$-subalgebra given by $\tilde{\Omega}'$. If $\gamma(\tilde{\Omega}) = \mho'$ (and vice versa) for an involution $\gamma \in \Gamma(4,20)$, the pair $\left((\Sigma, B, V, \tilde{\Omega}), (\Sigma', B', V', \tilde{\Omega}')\right)$ is called a **quantum mirror pair** and γ a **quantum mirror symmetry**.*

Remark 2.2 *Apart from classical symmetries, the above transformation can be interpreted as looking at the old fourplane $\tilde{\Omega} \oplus \mho$ from the viewpoint[13] of a new geometric interpretation $\mathbb{Z}\gamma v \oplus \mathbb{Z}\gamma v^0$.*

Obviously, this definition is not very restrictive. In particular, the trivial symmetry $\mathbb{1} \in \Gamma(4,20)$ is a quantum mirror symmetry.

This is caused by the fact that we almost completely ignored geometry: Over any point in the moduli space of $N = (4,4)$ theories one has an $\mathbb{S}^2 \times \mathbb{S}^2$ of $N = (2,2)$ theories corresponding to different choices of the $N = (2,2)$ subalgebra. In the geometrical context, such a choice is induced by the choice of complex structure, for which one only has one $\mathbb{S}^2$ to choose from. We thus arrive at the following definition:

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[13]The difference is analogous to the difference between rotating an object or the coordinate system used to measure its position.

Definition 2.3 *A quantum mirror pair* $\left((\Sigma, B, V, \tilde{\Omega}), (\Sigma', B', V', \tilde{\Omega}') \right)$ *is called a **geometric mirror pair** if both $\tilde{\Omega}$ and $\tilde{\Omega}'$ are preimages of complex structures $\Omega \subset \Sigma$ and $\Omega' \subset \Sigma'$. In this case, γ is called a **geometric mirror symmetry**.*

Proposition 2.4 *Let Ξ be a fourplane as above, $\mathbb{Z}v \oplus \mathbb{Z}v^0$ a geometric interpretation and $\gamma \in \Gamma(4, 20)$ an involution. Let $\xi \in \Xi$ be the vector defined by $\mathbb{Z}v \oplus \mathbb{Z}v^0$ as above and $\xi' \in \Xi$ the vector defined by the dual geometric interpretation $\mathbb{Z}\gamma v \oplus \mathbb{Z}\gamma v^0$. Then γ gives rise to a geometric mirror symmetry with respect to Ξ and $\mathbb{Z}v \oplus \mathbb{Z}v^0$, iff $\xi \perp \xi'$.*

Proof: It is clear, that for a geometric mirror pair, $\xi \in \mho$ and $\xi' \in \tilde{\Omega}$. For the other direction, choose $\xi^0 \in \langle \xi, \xi' \rangle^\perp \subset \Xi$ and define $\mho := \langle \xi, \xi^0 \rangle$. $\qquad \square$

Remark 2.5 *Proposition 2.4 implies, that neither the trivial symmetry nor the T-duality $v \leftrightarrow v^0$ are geometric mirror symmetries.*

If we want γ to still give rise to a geometric mirror symmetry when changing only the volume on either side of a given geometric mirror pair[14], we must have $v \perp \gamma v$:

$$\forall V : \xi = v^0 + B + (V - \frac{B^2}{2})v \perp \gamma v \ \Rightarrow v \perp \gamma v.$$

It is useful to also demand $v^0 \perp \gamma v$, which implies $\langle \gamma v^0, v \rangle = \langle \gamma v^0, \gamma^2 v \rangle = \langle v^0, \gamma v \rangle = 0$. Obviously, this additional condition can be fulfilled by an appropriate integer B-field shift. For this situation, we obtain

Corollary 2.6 *Let Ξ, $\mathbb{Z}v \oplus \mathbb{Z}v^0$ be as above and let $\mathbb{Z}w \oplus \mathbb{Z}w^0 \subset H^2(X, \mathbb{Z}) := (\mathbb{Z}v \oplus \mathbb{Z}v^0)^\perp \subset H^\star(X, \mathbb{Z})$ be a primitive hyperbolic sublattice. Assume $B \perp w$. Let $\gamma \in \Gamma(4, 20)$ be the involution given by exchanging $v \leftrightarrow w$ and $v^0 \leftrightarrow w^0$. Then γ gives rise to a geometric mirror symmetry.*

Proof: With $B \perp w$ we have $\xi \perp w$ and thus $\xi \in \Upsilon'$, where Υ' is given by the geometric interpretation $\mathbb{Z}w \oplus \mathbb{Z}w^0$. $\xi' \perp \Upsilon' \Rightarrow \xi' \perp \xi$. $\qquad \square$

Even with the latter definition, we do not have a complete analogy to mirror symmetry for Calabi-Yau threefolds. Note in particular, that deformations of Ξ fixing $\tilde{\Omega}$ also fix Ω, but the reverse is not true. Hence, the local splitting of the moduli space is not the same as distinguishing between deformations of the complex structure and deformations of Kähler form and B-field.

The deformations of Ξ fixing $\tilde{\Omega}$ are given by the deformation of V, the 20 deformations of B perpendicular to Ω and the 19 deformations of Σ fixing Ω. This almost looks like complexified Kähler deformations, the analogy being perfect when B is a $(1,1)$-form, i.e. $B \in \Omega^\perp$.

On the other hand, the deformations fixing $\mho$ contain the 38 deformations of Ω fixing its orthogonal complement in Σ, but also contain two deformations

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[14]i.e. we want to obtain another geometric mirror pair by changing either volume and adjusting the mirror partner accordingly

of $\tilde{\Omega}$ in the direction of $v^0 + B - (V + \frac{B^2}{2})v$, which render it useless as preimage of a complex structure.

In order to preserve such an interpretation, the corresponding deformation of Ξ has to be accompanied by a deformation of $\mho$. This effectively means replacing these two deformations by B-field shifts in the direction of Ω.

As remarked above, the analogy to the threefold case can be enhanced by demanding $B \in \Omega^\perp$. For any Ξ, this can be achieved by choosing $\Omega = \tilde{\Omega} \subset H^2(X, \mathbb{R})$. For $B \neq 0$ this uniquely determines $\tilde{\Omega} = \Upsilon \cap H^2(X, \mathbb{R}) = \Xi \cap H^2(X, \mathbb{R})$.

Such values of the B-field naturally arise in the context of algebraic K3 surfaces when restricting complexified Kähler forms on the ambient space to the K3 surface. By abuse of language, we will hence call such values of the B-field **algebraic**.

Remark 2.7 *In this context, a subtlety arises: It is always possible to choose Ω in such a way that the K3 surface becomes algebraic. To this end, one simply has to choose Ω to be orthogonal to any vector $\rho \in H^2(X, \mathbb{Z})$ with $\rho^2 > 0$ [Kod64] leading to a countable[15] infinity of such choices. In general though, none of these choices is compatible with the choice leading to an algebraic B-field.*

There has been some confusion in the literature regarding the notion of a nonalgebraic deformation. The above discussion shows that it does *make sense when imposing both conditions.*

Definition 2.8 *A geometric mirror pair $\left((\Sigma, B, V, \tilde{\Omega}), (\Sigma', B', V', \tilde{\Omega}') \right)$ is called an* **algebraic mirror pair** *if B and B' are algebraic B-fields (i.e. $\Omega = \tilde{\Omega}$ and $\Omega' = \tilde{\Omega}'$) and both (Σ, V, Ω) and (Σ', V', Ω') are algebraic K3 surfaces. In this case, the geometric mirror symmetry γ is called an* **algebraic mirror symmetry**.

The full analogy to the threefold case can be obtained by restricting to families of K3 sigma models where $\mho$ remains inside $(\gamma v)^\perp$, which means $B \perp \gamma v$ for $(\mathbb{Z}v \oplus \mathbb{Z}v^0) \perp (\mathbb{Z}\gamma v \oplus \mathbb{Z}\gamma v^0)$.

Definition 2.9 *Two families of K3 sigma models with chosen complex structures are called (quantum, geometric, algebraic)* **mirror families**, *if they are mapped to each other by a fixed (quantum, geometric, algebraic) mirror symmetry.*

2.3 Lattice polarized surfaces and orthogonal Picard lattices

We now consider K3 surfaces X with complex structure Ω, Kähler form ω and algebraic B-field B, i.e. $\Omega = \tilde{\Omega}$.

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[15]The set of choices is countable since the orthogonal complement of Σ in $H^2(X, \mathbb{Z})$ is negative definite.

As $H^2(X, \mathbb{Z})$ is an even selfdual lattice with signature $(3, 19)$, it is uniquely determined to be

$$H^2(X, \mathbb{Z}) = -E_8 \oplus -E_8 \oplus H \oplus H \oplus H, \tag{1}$$

where the scalar product of the lattice $-E_8$ is given by minus the Cartan matrix of E_8 and H is the hyperbolic lattice. Now define $\tilde{H}^2(X, \mathbb{Z})$ by $\tilde{H}^2(X, \mathbb{Z}) \oplus \tilde{H} := H^2(X, \mathbb{Z})$, where $\tilde{H}$ is one of the hyperbolic lattices from (1). For later convenience, we define $\hat{H} := H^0(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$.

By the Lefschetz theorem the Picard lattice of a K3 surface is given by $Pic(X) = H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$, the Picard number (its rank) is denoted by $\rho(X)$. Using the notations of the preceding section we therefore have

$$Pic(X) = H^2(X, \mathbb{Z}) \cap \Omega^\perp.$$

We now consider primitive nondegenerate sublattices $M_1 \subseteq M_2 \subset \tilde{H}^2(X, \mathbb{Z})$ with ranks ρ_1 and ρ_2, respectively.

For given sublattices M_1, M_2 we consider the family $K3_{M_1,M_2}$ of all sigma models on X with complex structure Ω, Kähler form ω and B-field B, subject to the following conditions[16]:

1. $M_2 \subseteq Pic(X) = \Omega^\perp \cap H^2(X, \mathbb{Z})$ $\qquad \Leftrightarrow \qquad \Omega \subset (M_2^\perp \oplus \tilde{H}) \otimes \mathbb{R}$

2. $B, \omega \in M_1 \otimes \mathbb{R}$ $\qquad\qquad\qquad\qquad \Leftrightarrow \qquad \mho \subset (M_1 \oplus \hat{H}) \otimes \mathbb{R}.$

Remark 2.10 *Whenever $B \neq 0$, Ω and $\mho$ are uniquely specified by the four-plane Ξ and the condition $\Omega \subseteq \Xi \cap H^2(X, \mathbb{R})$. Otherwise, the two conditions select a complex structure in Ξ.*

Remark 2.11 *If the family $K3_{M_1,M_2}$ is not empty, M_1 must obviously contain a vector with positive length squared. Hence, $K3_{M_1,M_2}$ consists of algebraic K3 surfaces with algebraic B-fields.*

Remark 2.12 *In the language of [Dol96] condition 1 says that we consider M_2-polarized K3 surfaces. Condition 2 poses additional constraints on the choice of B-field and Kähler form.*

Remark 2.13 *Splitting off $\tilde{H}$ is quite a loss of generality in comparison with the discussion in [Dol96]. It guarantees[17] (in the language of [Dol96]) the existence of a 1-admissible isotropic vector in $M_2^\perp \subset H^2(X, \mathbb{Z})$. It also implies [Dol96, Proposition 5.6] irreducibility of the moduli space of M_2-polarized K3 surfaces and enables us to ignore the orientation of Σ for the following discussions. The more general case of n-admissible vectors in [Dol96] obviously does not correspond to a mirror symmetry in our definition (or any definition demanding isomorphic conformal field theories) as such vectors do not define geometric interpretations.*

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[16]By $M_{1/2}^\perp \subset X$ we denote the orthogonal complement inside X, while $M_{1/2}^\perp$ alone always means $M_{1/2}^\perp \subset \tilde{H}^2(X, \mathbb{Z})$.

[17]In fact, it is equivalent: The existence of a 1-admissible vector by definition implies the existence of a primitive hyperbolic sublattice of $M_2^\perp \subset H^2(X, \mathbb{Z})$, which is unique up to isometry of $H^2(X, \mathbb{Z})$ [Dol96, Proposition 5.1].

We now consider the algebraic mirror symmetry γ given by exchanging v, v^0 with the generators of $\tilde{H}$. The corresponding mirror pairs are given by

$$\left((\langle \Omega, \omega \rangle_{\mathbb{R}}, B, V, \Omega), (\Sigma', B', V', \Omega') \right),$$

where Σ', B', V' are determined by $\gamma \Xi, v, v^0$ and $\Omega' = \gamma \mho$. Ω' indeed defines a complex structure since

$$\begin{aligned}
\Omega &\subseteq \Xi \cap \hat{H}^{\perp} \subset H^{\star}(X, \mathbb{Z}) \otimes \mathbb{R} \text{ and} \\
\mho &\subseteq \Xi \cap \tilde{H}^{\perp} \subset H^{\star}(X, \mathbb{Z}) \otimes \mathbb{R}.
\end{aligned}$$

As the projection of $\mho$ to $H^2(X, \mathbb{R})$ is spanned by B and ω,

$$Pic(X') \supseteq M_1^{\perp}$$

and

$$B', \omega' \in M_2^{\perp} \otimes \mathbb{R}.$$

The same argument works in both directions[18] and we obtain algebraic mirror symmetry for the families

$$K3_{M_1, M_2} \leftrightarrow K3_{M_2^{\perp}, M_1^{\perp}}$$

as a generalization of the mirror symmetry

$$K3_{M_1, M_1} \leftrightarrow K3_{M_1^{\perp}, M_1^{\perp}}$$

in [AM97].

Remark 2.14 *As long as we restrict ourselves to $M_1 = M_2$, generic members $X \in K3_{M_1, M_1}, X' \in K3_{M_1^{\perp}, M_1^{\perp}}$ of the two mirror symmetric families fulfill*

$$\rho(X) = \dim M_1 = \rho_1 = 20 - \dim M_1^{\perp} = 20 - \rho(X'),$$

the widely used landmark for mirror symmetric families of K3 surfaces. This formula is obviously not correct, if $M_1 \neq M_2$.

3 Picard lattices for toric K3 hypersurfaces

3.1 Toric preliminaries

We use (almost) standard notations as follows[19]. Let $N \cong \mathbb{Z}^3$ and $M = N^*$ be dual three dimensional lattices, $\Delta \subset M_{\mathbb{R}}$ a reflexive polyhedron as defined in [Bat94] and $\Delta^* \subset N_{\mathbb{R}}$ its polar dual. Let X_{Δ} denote the toric variety corresponding to some maximal crepant[20] refinement Σ of the normal fan $\mathcal{N}(\Delta)$ of

---

[18]We use the (more or less trivial) fact, that $((M_{\mathbb{R}})^{\perp} \cap \tilde{H}^2(X, \mathbb{Z})) \otimes \mathbb{R} = (M_{\mathbb{R}})^{\perp}$ for any sublattice $M \subset \tilde{H}^2(X, \mathbb{Z})$ $((M_{\mathbb{R}})^{\perp}$ is the kernel of a matrix with integer entries).

[19]For an introduction to toric geometry I recommend [Ful93, Oda85, Cox97]

[20]i.e. the resolution of singularities $X_{\Delta} = X_{\Sigma} \to X_{\mathcal{N}(\Delta)}$ preserves the canonical class.

Δ, i.e. the fan over some maximal triangulation of $\partial\Delta^* \cap N$ (see remark 3.1 below concerning the choice of triangulation). Let $(\mathbb{C}^\star)^3 \cong \mathbb{T}^3 \subset X_\Delta$ denote the three dimensional algebraic torus corresponding to the 0 dimensional cone in Σ. Let $\{D_i, i \in I\}$ denote the set of toric divisors of X_Δ corresponding to the rays $\{\rho_i \in \Sigma^{(1)} \subset \Sigma\}$. For each ρ_i let $n_i \in \rho_i$ be its primitive generator. Let $\mathcal{O}_{X_\Delta}(\sum_{i \in I} D_i)$ be the anticanonical line bundle on X_Δ. The set of zeroes $Z(\chi)$ of a generic global section $\chi \in \Gamma(\mathcal{O}_{X_\Delta}(\sum_{i \in I} D_i), X_\Delta)$ is called a **toric K3 hypersurface** in X_Δ. Using the holomorphic quotient construction of X_Δ the section χ can be identified with a homogenous polynomial

$$\chi = \sum_{m \in \Delta \cap M} a_m \prod_{n_i, i \in I} z_i^{\langle m, n_i \rangle + 1} \tag{2}$$

in the homogenous coordinate ring[21] S_Σ of X_Δ.

The following facts have been proven in [Bat94]:

1. For generic χ, $Z(\chi)$ is Σ-regular, i.e. the intersections $Z(\chi) \cap O_\sigma$ are either empty or smooth with codimension 1 in O_σ for all torus orbits O_σ corresponding to cones $\sigma \in \Sigma$.

2. For generic χ (and, as stated above, maximal triangulation of $\partial\Delta^*$), $Z(\chi)$ is a smooth two dimensional Calabi-Yau variety, i.e. a K3 surface.

Remark 3.1 *As toric divisors corresponding to points n_i on the (codimension 1) faces of Δ^* have empty intersection with the generic hypersurface (which was already proven in [Bat94] and can easily be checked using the formulas derived in section 3.2 below), one only has to maximally triangulate faces of Δ^* with codimension ≥ 2 to obtain a maximal crepant desingularization of the hypersurface $Z(\chi) \subset X_{\mathcal{N}(\Delta)}$. Since these are just the vertices and edges of Δ^*, this maximal triangulation is uniquely determined. This implies that the (usually numerous) cones of the secondary fan for X_Δ corresponding to smooth K3 surfaces combine to form a single geometric phase of the associated linear sigma model.*

For technical reasons it is nevertheless often convenient to work with a complete maximal triangulation of $\partial\Delta^*$. The intersection theory on the surface does not depend on the specific choice.

We now want to study the Picard lattice of the generic smooth toric K3 surface for given reflexive Polyhedron Δ. In [PS97] the formula

$$\rho(Z_\chi) = l(\Delta^*) - \dim\Delta - 1 - \sum_{\mathrm{codim}\Gamma^*=1} l^*(\Gamma^*) + \sum_{\mathrm{codim}\Gamma^*=2} l^*(\Gamma)l^*(\Gamma^*), \tag{3}$$

for the Picard number of the generic smooth toric K3 hypersurface $Z_\chi \subset X_\Delta$ was stated by reference to [Bat94]. In (3) Γ^* denotes a face of Δ^* of the given codimension, Γ the dual face of Δ, $l(\Delta^*)$ is the number of integer points in Δ^* and $l^*(\Gamma)$ is the number of integer points in the relative interior of Γ.

---

[21] For a discussion of this ring as well as the holomorphic quotient construction c.f. [Cox97].

This is both true and false. In [Bat94] the above formula was proven for the Hodge number $h^{1,1}(V)$ of a smooth Δ-regular toric Calabi-Yau hypersurface V with $\dim V \geq 3$. The case $V = K3$ with $h^{1,1}(K3) = 20$ was explicitly excluded. Nevertheless, the above formula is correct, but one has to be very careful about the conditions, under which it is true.

[Bat94, Theorem 4.4.2] only demands the Calabi-Yau hypersurface to be Δ regular, which can be assured by using just a subset of the linear system of global sections of the anticanonical bundle on X_Δ. In particular, one easily sees that the linear subsystem spanned by the sections corresponding to vertices of Δ suffices to carry through the Bertini type argument leading to Δ regularity of the Calabi-Yau hypersurface defined by the generic member. As we will shortly see, this does **not** suffice for validity of (3). Rather, the complete linear system is needed.

Example 3.2 *Consider the **Quartic line** of K3 hypersurfaces in $\mathbb{P}^3$ defined by $\chi = \sum_{i=1}^4 X_i^4 - 4\lambda \prod_{i=1}^4 X_i$, $\lambda \in \mathbb{C}$. An easy calculation shows, that all of these hypersurfaces except for λ a fourth root of unity (where the hypersurface has 16 node singularities inside the three dimensional open torus $\mathbb{T}^3$) are Δ-regular.*

The generic Picard number of a quartic in $\mathbb{P}^3$ as calculated by (3) is

$$\rho(Z_\chi) = 5 - 3 - 1 - 4 \cdot 0 + 6 \cdot 0 \cdot 3 = 1.$$

Using the algebraic automorphism group $\mathbb{Z}_4^2$ and methods developed in [Nik80], one can show that the generic Picard number of the quartic line is 19. This clearly shows that Δ regularity alone does not suffice.

What one *does* get from [Bat94] is a lower bound on the generic Picard number: When specialized to the K3 case, the exact sequence from the proof of [Bat94, Theorem 4.4.2] reads

$$0 \to H_c^{1,1}(Z_\chi \cap \mathbb{T}^3) \to H_c^{1,1}(Z_\chi) \to H_c^2(Z_\chi \backslash (Z_\chi \cap \mathbb{T}^3)) \to H_c^3(Z_\chi \cap \mathbb{T}^3) \to 0.$$

Now the irreducible components of $Z_\chi \backslash (Z_\chi \cap \mathbb{T}^3)$ clearly are divisors of Z_χ and because of $\text{rk}(H_c^3(Z_\chi \cap \mathbb{T}^3)) = \text{rk}(H_c^5(\mathbb{T}^3)) = 3$ and the above sequence the space of relations between them has dimension 3. The number of these components can be counted just as in higher dimension and for Δ-regular χ we obtain

$$\rho(Z_\chi) \geq \tilde{\rho}(\Delta) := l(\Delta^*) - 4 - \sum_{\text{codim}\Gamma^* = 1} l^*(\Gamma^*) + \sum_{\text{codim}\Gamma^* = 2} l^*(\Gamma)l^*(\Gamma^*).$$

In order to obtain an upper bound on the generic Picard number, we prove the following easy lemma:

Lemma 3.3 *The number of real deformations of complex structure fixing the Picard lattice $Pic \subset H^2(K3, \mathbb{Z}) \cong \mathbb{Z}^{3,19}$ of a given K3 variety V with $\text{rk}Pic = \rho$ is $D = 2(20 - \rho)$.*

Proof: The deformations are just the deformations of the oriented positive definite two-plane $\Omega \subset Pic^{\perp} \subset \mathbb{R}^{3,19}$ discussed in the preceding sections by elements of $H^{1,1}(V, \mathbb{R})$, i.e. by elements of $\mathbb{R}^{3,19}$ perpendicular to Ω. With $a \in \{0, 1\}$ we obtain[22]

$$D = \dim so(3 - a, 19 - \rho + a) - \dim so(1 - a, 19 - \rho + a) - \dim so(2) = 2(20 - \rho).$$

$\square$

Corollary 3.4 *Let F be a family of K3 surfaces with k continuous real deformation parameters for the complex structure. Then the Picard number of a generic member $V \in F$ cannot exceed $20 - \lfloor k/2 \rfloor$.*

Remark 3.5 *The condition* generic *in Corollary 3.4 is stronger than usual in algebraic geometry. It is here used in the same way as the generic element of $\mathbb{R}$ is irrational.*

Now, the first term in (3.1) just represents deformations of the complex structure by deformations of the defining polynomial. Since $h^{1,1}(Z_\chi) = 20$, we can use Lemma 3.4 with $k = 2 \dim H_c^{1,1}(Z_\chi \cap \mathbb{T}^3) = 2(20 - \tilde{\rho}(\Delta))$. For the K3 hypersurface Z_χ with χ a generic member of the complete linear system we thus obtain

$$\tilde{\rho}(\Delta) \leq \rho(Z_\chi) \leq 20 - \frac{2}{2}(20 - \tilde{\rho}(\Delta)) = \tilde{\rho}(\Delta).$$

Remark 3.6 *When considering families of K3 surfaces allowing all deformations of the complex structure preserving some original Picard lattice $Pic(V_0)$, points with enlarged Picard lattice for any $\mathrm{rk}Pic(V) > \mathrm{rk}Pic(V_0)$ are dense. Dimensional counting shows that this is the case for families of toric K3 hypersurfaces.*

We now want to calculate not only the rank, but the complete Picard lattice of a toric K3 hypersurface $Z(\chi)$, which amounts to calculating the Chow group $A^1(Z(\chi))$ and the intersection pairing

$$\# : A^1(Z(\chi)) \times A^1(Z(\chi)) \dot{\to} A^2(Z(\chi)) \cong \mathbb{Z}.$$

One part of this Chow group is easily calculated and stems from the Chow group $A^1(X_\Delta)$ of the toric variety itself. This Chow group is generated by the toric divisors of X_Δ subject to the linear relations given by the exact sequence

$$0 \to M \xrightarrow{\alpha} \bigoplus_{i \in I} \mathbb{Z}D_i \to A^1(X_\Delta) \to 0, \tag{4}$$

where $\alpha : M \ni m \mapsto \sum_{i \in I} \langle m, n_i \rangle D_i$ and $\langle ., . \rangle$ is the natural pairing between M and N. As we have chosen a maximal triangulation of $\partial \Delta^*$, three pairwise different toric divisors D_i, D_j, D_k yield

$$D_i \cdot D_j \cdot D_k = \begin{cases} 1 & \text{if } \rho_i, \rho_j, \rho_k \text{ are contained in a single cone } \sigma \in \Sigma \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

---
[22]$a = 1$ for algebraic V

We are not interested in this Chow group itself, but rather in its image under the homomorphism $i^\star : A^\star(X_\Delta) \to A^\star(Z(\chi))$, where i denotes the embedding $Z(\chi) \to X_\Delta$. For future use we define $Pic_{\text{tor}}(\Delta) := i^\star A^1(X_\Delta)$ for a generic section χ.

For the intersections $\#(i^\star D_i, i^\star D_j)$ we obtain $\#(i^\star D_i, i^\star D_j) = i^\star D_i \cdot i^\star D_j = i_\star(i^\star D_i \cdot i^\star D_j) = i_\star i^\star(D_i \cdot D_j) = Z(\chi) \cdot D_i \cdot D_j$.

As already mentioned above, divisors D_i corresponding to points n_i in the interior of codimension 1 faces of Δ^* are in the kernel of $i^\star$. This is easily checked using $Z(\chi) = \sum_i D_i$ in $A^1(X_\Delta)$: Let n_i lie in the interior of the facet Γ^* of Δ^* given by the inner face normal $m \in M$ ($\Gamma^* = \{n \in \Delta^* | \langle m, n \rangle = -1\}$). Then

$$D_i \cdot Z(\chi) = D_i \cdot \sum_{j \in I} D_j = D_i \cdot \sum_{n_j \in \Gamma^*} D_j.$$

Now according to (4) we have

$$0 = \sum_{j \in I} \langle m, n_j \rangle D_j$$

in $A^1(X_\Delta)$ and therefore

$$D_i \cdot Z(\chi) = D_i \cdot \left(\sum_{n_j \in \Gamma^*} D_j + \sum_{j \in I} \langle m, n_j \rangle D_j \right) = D_i \cdot \sum_{n_j \notin \Gamma^*} \langle m, n_j \rangle D_j = 0.$$

Hence, we will happily ignore these divisors from now on. The rest of $A^1(X_\Delta)$ maps injectively to $A^1(Z(\chi))$ and the intersection matrix can be calculated by variations of the above theme. Though this has already been done in[23] [PS97], for completeness of the exposition I will repeat the calculation in the following section[24].

3.2   Calculation of the toric Picard lattice Pic_{tor}

We first turn to the intersection of two different divisors $i^\star D_1$ and $i^\star D_2$. If either n_1 or n_2 lies in the interior of a facet of Δ^*, the intersection obviously vanishes. The same holds for n_1 and n_2 which lie on different edges of Δ^*. If the different edges are not borders of a common facet, this is obvious. Let us now assume n_1 and n_2 belong to a common cone in Σ. In this case, they obviously belong to exactly two common cones. We denote the corresponding third generators by n_3 and n_4, which lie on (not necessarily different) facets of Δ^* given by inner face

---

[23]The second author of [PS97] does not want to be cited in this context, assumingly because derivation of these formulas is just an application of standard formulas from the literature. Even the basics necessary to calculate the full intersection matrix as I will do in section 3.3 are already contained in [Bat94].

[24]In particular my formula for the self-intersection number of a divisor belonging to a vertex of $\Delta^\star$ can be used in all cases and is thus much better suited to be used in a computer program than the formulas derived in [PS97].

normals m_3 and m_4, i.e. $\langle m_3, n_3 \rangle = -1$ and $\langle m_4, n_4 \rangle = -1$. For the intersection we obtain

$$
\begin{aligned}
i^\star D_1 \cdot i^\star D_2 &= \sum_i D_1 \cdot D_i \cdot D_2 & (5) \\
&= D_1 \cdot D_1 \cdot D_2 + D_1 \cdot D_2 \cdot D_2 + D_1 \cdot D_3 \cdot D_2 + D_1 \cdot D_4 \cdot D_2
\end{aligned}
$$

Because of (4) we have

$$
0 = \sum_i \langle m_3, n_i \rangle D_i \Rightarrow D_1 = -D_2 - D_3 + \langle m_3, n_4 \rangle D_4 + \dots, \quad (6)
$$

where $D_1 \cdot (\dots) \cdot D_2 = 0$. If we insert this into the first term in (5) (for one D_1 only), we obtain

$$
\begin{aligned}
i^\star D_1 \cdot i^\star D_2 &= D_1 \cdot D_2 \cdot D_4 + \langle m_3, n_4 \rangle D_1 \cdot D_2 \cdot D_4 \\
&= (1 + \langle m_3, n_4 \rangle) D_1 \cdot D_2 \cdot D_4 \\
&= \langle m_3 - m_4, n_4 \rangle. & (7)
\end{aligned}
$$

If now n_1 and n_2 lie on different borders of the same facet, $m_3 = m_4$ and the intersection vanishes[25].

For two points n_1 and n_2 on (not necessarily in the interior of) the same edge θ^* of Δ^* the intersection can only be nonzero if the two points are neighboring. In this case and for any maximal triangulation $\partial \Delta^*$ they share two common cones and m_3 and m_4 are the inner face normals of the two facets which intersect in θ^*.

Because of $\langle m_3 - m_4, n_1 \rangle = \langle m_3 - m_4, n_2 \rangle = 0$ and as $\{n_1, n_2, n_4\}$ form a basis of N, (7) is just the integer length of $m_3 - m_4$, i.e. the length[26] $l(\theta)$ of the edge θ of Δ dual to $\theta^\star$:

$$
i^\star D_1 \cdot i^\star D_2 = l(\theta). \quad (8)
$$

We now turn to the self-intersection number for $i^\star D_1$ where n_1 lies in the interior of some edge θ^*. Denote its two neighboring points on θ^* by n_2 and n_3. Again let one of the neighboring facets be given by inner face normal m_3. Using (6) and (8) we obtain

$$
\begin{aligned}
i^\star D_1 \cdot i^\star D_1 &= D_1 \cdot Z(\chi) \cdot \sum_{i \neq 1} \langle n_i, m_3 \rangle D_i \\
&= -D_1 \cdot Z(\chi) \cdot (D_2 + D_3) + 0 \\
&= -2\, l(\theta). & (9)
\end{aligned}
$$

---

[25]This is just what one expects, since intersections on the K3 surface should not depend on the triangulation of the facets.

[26]To avoid confusion, we should note that $l(\theta)$ is *not* the number of integer points in θ (in contrast to the notation in (3)) but rather #(integer points in θ) $= l(\theta) + 1 = l^\star(\theta) + 2$.

For the remaining case of a vertex n_1 of Δ^* let one of the facets containing n_1 be given by m_1, i.e. $\langle m_1, n_1 \rangle = -1$. We then obtain

$$
\begin{aligned}
i^\star D_1 \cdot i^\star D_1 &= D_1 \cdot Z(\chi) \cdot \sum_{i \neq 1} \langle m_1, n_i \rangle \, D_i \\
&= D_1 \cdot Z(\chi) \cdot \left(\sum_{i=2}^{k} \langle m_1, n_i \rangle \, D_i \right) \\
&= \sum_{j=2}^{k} \langle m_1, n_j \rangle \, i^\star D_1 \cdot i^\star D_j.
\end{aligned}
\tag{10}
$$

The $i^\star D_1 \cdot i^\star D_j$ have already been calculated using (8).

3.3 Calculation of the complete Picard lattice

If the **toric correction term** $\delta = \sum_{\mathrm{codim}\Gamma^*=2} l^*(\Gamma) l^*(\Gamma^*)$ in (3) vanishes, we have $Pic(\Delta) := Pic(Z(\chi)) = Pic_{\mathrm{tor}}(Z(\chi))$ for generic χ. Now let us assume $\delta \neq 0$ and let D_i be a toric divisor corresponding to a point n_i in the interior of an edge θ^*, for which the dual edge θ also contains interior integer points. Then the intersection of D_i with the K3 hypersurface splits into $l(\theta)$ disjoint so-called **nontoric divisors**. This well known fact is most easily seen by using the homogenous polynomial description for χ. When restricting to the divisor under consideration, this polynomial reduces to a polynomial in one variable, the zeroes of which determine the (for general χ) disjoint nontoric divisors: Using the description (2) and restricting to D_i, i.e. setting $z_i = 0$, yields

$$
\begin{aligned}
\chi|_{z_i=0} &= \sum_{m \in \Delta \cap M} a_m \prod_{n_j, j \in I} z_j^{\langle m, n_j \rangle + 1} \\
&= \sum_{m \in \theta \cap M} a_m \prod_{n_j \notin \theta^*} z_j^{\langle m, n_j \rangle + 1} \\
&= a_{m_0} \prod_{n_j \notin \theta^*} z_j^{\langle m_0, n_j \rangle + 1} \sum_{k=0}^{l(\theta)} \frac{a_{m_k}}{a_{m_0}} r^k
\end{aligned}
$$

where $m_0, \ldots, m_{l(\theta)}$ denote the integer points along θ and r is the rational function on X_Δ defined by $r = \prod_{j \in I} z_j^{\langle m_1 - m_0, n_j \rangle}$.

Consider such a divisor splitting into nontoric divisors as

$$
i^\star D_1 = \sum_j \tilde{D}_1^{(j)}.
$$

The embedding of each nontoric component into X_Δ is given by the intersection of D_1 with the set $\{r = z_j\}$, where r is defined as above and z_j is the corresponding zero of the polynomial $\chi|_{z_i=0}$ considered as a function of r. This description implies, that

$$
\forall j, k : i_\star \tilde{D}_1^{(j)} = i_\star \tilde{D}_1^{(k)},
\tag{11}
$$

since the rational functions $r - z_j$ and $r - z_k$ only differ by a constant and the difference of their sets of zeroes consequently is just the principal divisor defined by the rational function $\frac{r-z_j}{r-z_k}$.

Using the explicit description it is clear that

$$i^\star D_1 \cdot i^\star D_2 = 0 \;\Rightarrow\; \forall j: \tilde{D}_1^{(j)} \cdot i^\star D_2 = 0.$$

If $i^\star D_2$ also splits into several nontoric divisors, it is also clear that

$$i^\star D_1 \cdot i^\star D_2 = 0 \;\Rightarrow\; \forall j, k: \tilde{D}_1^{(j)} \cdot \tilde{D}_2^{(k)} = 0.$$

As the restricted polynomial is the same for all toric divisors corresponding to points in the interior of the same edge θ^*, for neighboring points n_1, n_2 one has:

$$\tilde{D}_1^{(j)} \cdot \tilde{D}_2^{(k)} = 0 \text{ whenever } j \neq k. \tag{12}$$

Since $i_\star$ maps the class of a point to the class of a point, using (11) one can deduce

$$\begin{aligned}
\forall j, k: \; \tilde{D}_1^{(j)} \cdot i^\star D_2 &= i_\star(\tilde{D}_1^{(j)} \cdot i^\star D_2) \\
&= D_2 \cdot i_\star \tilde{D}_1^{(j)} \\
&= D_2 \cdot i_\star \tilde{D}_1^{(k)} \\
&= i_\star(\tilde{D}_1^{(k)} \cdot i^\star D_2) = \tilde{D}_1^{(k)} \cdot i^\star D_2
\end{aligned}$$

and therefore

$$\tilde{D}_1^{(j)} \cdot i^\star D_2 = \frac{1}{l(\theta)} i^\star D_1 \cdot i^\star D_2. \tag{13}$$

Because of (12) we thus obtain

$$\begin{aligned}
\tilde{D}_1^{(j)} \cdot \tilde{D}_1^{(k)} &= \delta_{j,k}\, \tilde{D}_1^{(j)} \cdot i^\star D_1 \\
&= \delta_{j,k} \frac{1}{l(\theta)} i^\star D_1 \cdot i^\star D_1
\end{aligned} \tag{14}$$

for the intersection numbers of nontoric divisors belonging to the same toric divisor. For the intersections of the nontoric components of divisors belonging to neighboring points n_1, n_2 on an edge of Δ^* we finally obtain

$$\begin{aligned}
\tilde{D}_1^{(j)} \cdot \tilde{D}_2^{(k)} &= \delta_{j,k}\, \tilde{D}_1^{(j)} \cdot i^\star D_2 \\
&= \delta_{j,k} \frac{1}{l(\theta)} i^\star D_1 \cdot i^\star D_2.
\end{aligned} \tag{15}$$

4   Main result and computer proof

We are now ready to state our main result. First note that by using a toric hypersurface as string compactification space we imply taking the metric and B field as the restriction of corresponding data on the ambient toric variety[27].

---

[27]In e.g. Witten's linear sigma model approach [Wit93] this restriction is implicitly contained.

Proposition 4.1 *Let Δ be a reflexive three dimensional polyhedron. The family of smooth toric K3 hypersurfaces in X_Δ (together with complexified Kähler form) is an analytically open subset of the family $K3_{Pic_{tor}(\Delta), Pic(\Delta)}$ as defined in section 2.3. The following equations hold:*

1. *$Pic_{\mathrm{tor}}(\Delta) = (Pic(\Delta^\star))^\perp \subset \tilde{H}^2(X, \mathbb{Z})$ and*

2. *$Pic_{\mathrm{tor}}(\Delta^\star) = (Pic(\Delta))^\perp \subset \tilde{H}^2(X, \mathbb{Z})$*

The first part of proposition 4.1 follows from the definition (c.f. section 2.3) and counting dimensions. Dimensional counting also shows, that the set of all twoplanes Ω corresponding to smooth toric K3 hypersurfaces in X_Δ (i.e. obtainable by polynomial deformations) is a Zariski open subset of the set of twoplanes in $Pic(\Delta)^\perp$. The restriction to the analytically open interior of a bounded domain only occurs for the twoplanes $\mho \subset Pic_{tor}(\Delta) \otimes \mathbb{R}$. Hence, we obtain

Corollary 4.2 *Analytically open subsets of the families of smooth K3 hypersurfaces in X_Δ and X_{Δ^*} are mirror dual.*

Remark 4.3 *The above restriction to open subsets can be lifted by consideration of additional phases of the corresponding theories. These phases are closely correlated with the secondary fans of Δ and $\Delta^\star$. Since at least some of the occuring phases cannot be described as nonlinear K3 sigma models, this is clearly outside the scope of this paper.*

What remains to prove is

$$Pic_{\mathrm{tor}}(\Delta) = (Pic(\Delta^\star))^\perp \subset \tilde{H}^2(X, \mathbb{Z}). \tag{16}$$

As $\tilde{H}^2(X, \mathbb{Z}) = -E_8 \oplus -E_8 \oplus H \oplus H$ is an even selfdual lattice just like $H^2(X, \mathbb{Z})$ itself, the following tools will be useful:

Definition 4.4 *Let Γ be an even, nondegenerate lattice (i.e. $\forall x \in \Gamma : \langle x, x \rangle \in 2\mathbb{Z}$ and $\langle x, y \rangle = 0 \, \forall y \in \Gamma \Rightarrow x = 0$). One then has a natural embedding of Γ into its dual lattice $\Gamma^\star$ and because of $\Gamma^\star \subset \Gamma \otimes \mathbb{Q}$ the scalar product on Γ can be uniquely extended to $\Gamma^\star$. The **discriminant group** of Γ is defined to be*

$$G_D(\Gamma) := \Gamma^\star / \Gamma.$$

Obviously, the discriminant group is finite and abelian. Using the scalar product on Γ one can define a quadratic form on the discriminant group:

Definition 4.5 *Let Γ be as in definition 4.4. On the discriminant group $G_D(\Gamma)$ one defines a quadratic form $q : G_D(\Gamma) \to \mathbb{R} \bmod 2\mathbb{Z}$ as follows: Let $x \in \Gamma^\star$ be a representative of $\bar{x} \in G_D(\Gamma)$. We set*

$$q(\bar{x}) := \langle x, x \rangle \bmod 2\mathbb{Z}.$$

q is well defined, because for $x \in \Gamma^\star$ and $y \in \Gamma$

$$\langle x + y, x + y \rangle - \langle x, x \rangle = 2 \langle x, y \rangle + \langle y, y \rangle \in 2\mathbb{Z}.$$

*q is called the **discriminant form**.*

Lemma 4.6 (**[Nik80]**) *Let $L \subset \Gamma$ be a primitive sublattice of an even, selfdual lattice Γ. Let $L^\perp$ denote its orthogonal complement and assume $L \cap L^\perp = \{0\}$. Then $L^\star/L \cong (L^\perp)^\star/L^\perp$ and the quadratic forms only differ by sign. Conversely, if $L, L^\perp$ are nondegenerate even lattices with the same discriminant form up to sign (we will denote the isomorphism by γ), then $L^\perp$ is the orthogonal complement of L in*

$$\Gamma := \left\{ (l, l') \in L^\star \oplus (L^\perp)^\star \mid \gamma(\bar{l}) = \bar{l}' \right\},$$

where $\bar{l}$ denotes the image of l under the quotient map $L^\star \to L^\star/L$.

Using arguments as in [Bat94] one can show that the objects in (16) have the right dimensions[28]. However, this is of course not sufficient to prove (16). The formulas given in section 3 are sufficient to calculate the lattices in (16) for any given reflexive Δ, but unfortunately do not allow for a general proof.

Here, the known classification of all three dimensional reflexive polyhedra in [KS98] comes to the rescue. As there are only 4319 reflexive polyhedra in three dimensions, one can show (16) for each reflexive Δ seperately.

Although the necessary calculations are simple enough to be done by hand, the number of polyhedra to check suggest delegating this work to a computer, in particular since the used classification has also been done by computer.

For the sake of using lemma 4.6, we can still prove the following[29]

Lemma 4.7 *Let Δ be a reflexive polyhedron, $\dim \Delta = 3$. Then both $Pic(\Delta)$ and $Pic_{\mathrm{tor}}(\Delta)$ are even nondegenerate lattices.*

Proof: As sublattices of $H^2(Z(\chi), \mathbb{Z})$ both lattices are even. $Pic(\Delta)$ is nondegenerate due to Poincaré duality on the K3 surface. Let $Pic_{\mathrm{tor}}(\Delta) \ni x = i^\star \tilde{x},\ \tilde{x} \in A^1(X_\Delta)$.

Suppose $\forall y \in Pic_{\mathrm{tor}}(\Delta) : x \cdot y = 0 \iff \forall \tilde{y} \in A^1(X_\Delta) : \tilde{x} \cdot Z(\chi) \cdot \tilde{y} = 0$. Due to Poincaré duality on the smooth toric variety X_Δ, $\tilde{x} \cdot Z(\chi) = 0 \Rightarrow \tilde{x} \in \ker i^\star \Rightarrow x = 0$. $\qquad\square$

It now suffices to perform the following calculation for each Δ from the classification list:

1. Calculate $Pic_{\mathrm{tor}}(\Delta)$ and $Pic(\Delta^\star)$ using the formulas in sections 3.2 and 3.3.

2. Calculate the discriminant groups and forms. In this step one uses finiteness of the discriminant group, which is generated by the equivalence classes of the generators of the dual lattice (which we denote by $L^\star$). $L^\star/L$ can then be constructed by repeatedly adding the equivalence classes of these generators. To this end one starts with $0 \in L^\star$, which yields the unit

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[28]Use $\mathrm{rk}(i^\star A^1(X_\Delta)) + 3 = \dim H^{1,1}(X_{\Delta^\star}^a) = \#\{$integer points on faces of Δ with codim $\geq 2\}$ and the exact sequence from the proof of theorem 4.4.2.

[29]This fact can easily be checked while explicitly calculating the lattices – nevertheless a general proof is nicer.

of the discriminant group. One now constructs a list of all group elements by adding all generators of $L^\star$ to representatives of all group elements already contained in the list. The newly constructed lattice points are then transformed into the fundamental domain of the sublattice $L \subset L^\star$ and added to the list, if they are not already contained. Obviously the algorithm stops when representatives for all elements of the discriminant group are contained in the list. After this calculation, the group multiplication table and the quadratic form are readily calculable.

3. Split the discriminant groups into their cyclic factors[30] $\mathbb{Z}_{p^q}$ with p prime. To this end one first determines the length of the orbits of all group elements, thereby splitting the discriminant group into factors $G_p := \bigoplus_k \mathbb{Z}_{p^{q_k}}$ for the pairwise different occurring prime numbers p. Beginning with orbits of maximal length, each G_p is then split into its cyclic factors $\mathbb{Z}_{p^q}$. If the discriminant groups under consideration split into (modulo reordering) different cyclic factors, we can stop the calculation as the discriminant groups are different.

4. As the embedding of the cyclic factors is not uniquely determined, one now constructs an exhaustive list of isomorphism candidates by trying to map the generators of the cyclic factors of the first discriminant group (as chosen during step 3 above) to elements of the second one with equal orbit length, thereby defining isomorphisms of cylclic subgroups. If the second discriminant form evaluated on the images differs from the first one evaluated on the preimages only by sign, one finally checks the constructed collection of isomomorphisms of cyclic subgroups for being an isomorphism of the discriminant groups by constructing the whole image and comparing multiplication tables[31]. If an isomorphism preserving the discriminant form (up to sign) is found, we are done. If all possible mappings are checked and such an isomorphism is not found, (16) must be false.

For cases with many isomorphic cyclic factors this step takes the most computing time.

The described algorithm was implemented using C++ and (using an 800 MHz PC) applied to all 4319 reflexive polyhedra from the classification list. After roughly two hours of computing time all discriminant groups were calculated and for all cases suitable isomorphisms were successfully determined. This proves proposition 4.1 by explicit calculation.

A list of the found discriminant groups and forms can be found at [Roh].

Remark 4.8 *Specifying the matrix of the discriminant form on the generators of the discriminant group completely determines the full discriminant form, because evaluating the scalar product on representatives of the group elements is*

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[30]This is possible for any finite abelian group. One possible proof of this well known fact consists of studying the steps of the algorithm presented in this section.

[31]The order of the tests is of course not mandatory – it just makes the algorithm faster to perform the easiest checks first.

well defined modulo $\mathbb{Z}$ (the offdiagonal elements of the matrices in the above list are therefore only well defined modulo $\mathbb{Z}$, not modulo $2\mathbb{Z}$!). This nevertheless allows calculating the squares of all other group elements:

If we choose a basis for the even nondegenerate lattice L and the corresponding dual basis for $L^\star$ and denote the matrix of the scalar product on L by g, the embedding $L \hookrightarrow L^\star$ is given by the matrix g and the induced scalar product on $L^\star$ is given by g^{-1}. We write $\{l_k\}$ for an arbitrarily chosen collection of representatives of generators of $L^\star/L$.

Modulo $2\mathbb{Z}$ we then have:

$$\forall a_i \in \mathbb{Z} : g^{-1}(\sum_k a_k l_k, \sum_k a_k l_k) = \sum_k a_k^2 g^{-1}(l_k, l_k) + \sum_{j \neq k} 2a_j a_k g^{-1}(l_j, l_k).$$

Hence, the scalar products between the generators only have to be defined modulo $\mathbb{Z}$ in order to reconstruct the complete quadratic form. The latter is true because modulo $2\mathbb{Z}$ $\forall l' \in L$:

$$g^{-1}(l_j + gl', l_k) = g^{-1}(l_j, l_k) + g^{-1}(gl', l_k)$$

and $g^{-1}(gl', l_k) \in \mathbb{Z}$.

Remark 4.9 *The transformation into the fundamental domain in step 2 of the algorithm can be done as follows:*

Let a matrix Λ define an embedding of some sublattice of rank d into $\mathbb{Z}^d$ (here $\Lambda = g$). A lattice point $x \in \mathbb{Z}^d$ lies in the fundamental domain given by Λ, if and only if (over $\mathbb{Q}$)
$$\Lambda^{-1}x \in [0,1)^d.$$

Hence, transforming any lattice point x into the fundamental domain can be done by setting

$$\bar{x} := \Lambda(\Lambda^{-1}x \bmod \mathbb{Z}) = x - \Lambda\lfloor \Lambda^{-1}x \rfloor \in \mathbb{Z}^d.$$

This calculation can be done using only integers, if one calculates the inverse of Λ (e.g. using Gaussian elimination) over $\mathbb{Z}$, i.e. $\Lambda^{-1} = \frac{\tilde{\Lambda}}{\lambda}$, $\tilde{\Lambda}$ and λ integer. Then

$$\bar{x} = \Lambda(\frac{\tilde{\Lambda}}{\lambda}x \bmod \mathbb{Z}) = \frac{1}{\lambda}\Lambda(\tilde{\Lambda}x \bmod \lambda\mathbb{Z}).$$

Remark 4.10 *Some nice explicit examples calculated using the methods in a preliminary version of this paper can be found (embedded into a larger context) in [Wen01].*

Acknowledgements

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